# 4. TROSTRUKI INTEGRAL

## PREGLED TEORIJE

**4.1.** Neka je $V$ ograničena oblast u prostoru čiji je rub po dijelovima glatka površ $S$. Ako je $V=\bigcup_{i=1}^{n} V_i$, gdje su $V_i$ podoblasti oblasti $V$ sa po dijelovima glatkim rubovima $S_i$ $(i=1, 2, \ldots, n)$ koje mogu imati zajedničke samo rubne tačke, tada se $V_1, \ldots, V_n$ zove podjela oblasti $V$. U tom slučaju oblast $V$ ima zapreminu $v(V)$, a podoblast $V_i$ zapreminu

$$\Delta V_i = v(V_i) \quad (i=1, 2, \ldots, n) \tag{1}$$

i vrijedi

$$v(V)=\sum_{i=1}^{n} \Delta V_i. \tag{2}$$

**4.2.** Neka je na $V$ zadana ograničena funkcija

$$f=f(T)=f(x, y, z) \quad (T(x, y, z)\in V). \tag{3}$$

Tada svakoj podjeli $V_1, V_2, \ldots, V_n$ oblasti $V$ i svakom izboru tačaka $T_i\in V_i$ $(i=1, 2, \ldots, n)$ odgovara integralna suma

$$\sigma_n=\sum_{i=1}^{n} f(T_i)\cdot \Delta V_i \tag{4}$$

funkcije $f(T)$. Ako postoji konačan limes

$$I=\lim_{\max d(V_i)\to 0} \sigma_n, \tag{5}$$

tada se kaže da je funkcija (3) integrabilna na $V$ i piše se

$$I=\iiint_V f(T)\, dV=\iiint_V f(x, y, z)\, dxdydz. \tag{6}$$

Izraz (6) zove se trostruki integral funkcije $f(T)$ po oblasti $V$. U relaciji (5) označen je sa $d(V_i)$ dijametar oblasti $V_i$, tj.

$$d(V_i)=\sup_{T, T'\in V} |\overrightarrow{TT'}|,$$

a smisao te relacije je sljedeći: za svako zadano $\varepsilon>0$ treba do postoji $\delta=\delta(\varepsilon)>0$ sa osobinom

$$\max d(V_i)<\delta \Rightarrow |I-\sigma_n|<\varepsilon, \tag{8}$$

za svaku podjelu oblasti $V$ i za svaki izbor tačaka $T_i\in V_i$.

**4.3.** Ako je funkcija (3) po dijelovima neprekidna na $V$, tada je funkcija (3) sigurno integrabilna na $V$. Specijalno je na $V$ integrabilna svaka konstantna funkcija $k$ i vrijedi

$$\iiint\limits_V k\,dV = k \cdot v(V). \tag{9}$$

Osim toga za svake dvije funkcije $f(T)$ i $g(T)$ integrabilne na $V$ vrijede relacije analogne onima iz tačke 2.5. navedenim za integral po luku.

**4.4.** Neka je oblast $V$ ograničena po dijelovima glatkim površima

$$z = z_1(x, y),\ z = z_2(x, y)\ ((x, y) \in G) \tag{10}$$

i dijelom cilindrične površi $g(x, y) = 0$, čije su izvodnice paralelne sa $z$-osom, a direktrisa joj je po dijelovima glatki rub

$$C : g(x, y) = 0,\ z = 0 \tag{11}$$

oblasti $G$. Ako osim toga postoji integral (6) i integral

$$I(x, y) = \int\limits_{z_1(x,y)}^{z_2(x, y)} f(x, y, z)\,dz\ (\forall\,(x, y) \in G), \tag{12}$$

tada postoji i integral

$$\iint\limits_G I(x, y)\,dxdy = \iint\limits_G \Big(\int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz\Big)\,dxdy \tag{13}$$

i integral (13) podudara se sa integralom (6), tj. vrijedi

$$\iiint\limits_V f(x, y, z)\,dxdydz = \iint\limits_G \Big(\int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz\Big)\,dxdy. \tag{14}$$

Sve pretpostavke ispunjene su, ukoliko je funkcija $f(x, y, z)$ neprekidna na $V$.

Relacija (14) pokazuje kako se uz učinjene pretpostavke trostruki integral po oblasti $V$ svodi na uzastopno računanje jednog jednostrukog i jednog dvostrukog integrala. Ako su za dvostruki integral koji se tu pojavljuje ispunjene pretpostavke iz tačke 3.5., tada se on računa uzastopnim računanjem dvaju dvostrukih integrala. U tom slučaju, dakle, računanje trostrukog integrala svodi se na uzastopno računanje triju jednostrukih.

**4.5.** Neka su na oblasti $V$ zadane diferencijabilne funkcije:

$$u = u(x, y, z),\ v = v(x, y, z),\ w = w(x, y, z) \tag{15}$$

koje oblast $V$ biunivoko preslikavaju na neki skup $V'$ prostora $O'uvw$, tj. za koje postoje jedinstvene funkcije

$$x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w) \tag{16}$$

definisane na $V'$ sa osobinom

$$x = x(u(x, y, z),\ v(x, y, z),\ w(x, y, z)), \ldots\ (\text{na } V). \tag{17}$$

Tada su i funkcije (16) diferencijabilne i imaju Jakobijan

$$J = \frac{D(x, y, z)}{D(u, v, w)} \neq 0 \text{ (na } V'). \tag{18}$$

Osim toga $V'$ je također ograničena oblast sa po dijelovima glatkim rubom i

$$v(V) = \iiint_{V'} |J| \, du dv dw.$$

Za funkciju $f(x, y, z)$ neprekidnu na $V$ vrijedi

$$\iiint_V f(x, y, z) \, dx dy dz = \iiint_{V'} f(x(u, v, w), \ldots). \; |J| \, du dv dw. \tag{20}$$

Relacija (20) kazuje kako se uz određene pretpostavke vrši smjena (17) varijabli u trostrukom integralu.

**4.6.** Ako je u oblasti $V$ raspoređena masa date prostorne gustoće, tada se ukupna masa, koordinate težišta, moment inercije i privlačna sila kojom oblast $V$ privlači materijalnu tačku $T_0(x_0, y_0, z_0)$ u kojoj je skoncentrisana masa $m_0$ računaju analogno slučaju linijskog i dvostrukog integrala.

**4.7.** Trostruki integral

$$I = \iiint_V \vec{F}(T) \, dV \tag{21}$$

vektorske funkcije

$$\vec{F}(T) = F_1(x, y, z)\vec{i} + F_2(x, y, z)\vec{j} + F_3(x, y, z)\vec{k} \tag{22}$$

po oblasti $V$ definiše se i računa slično kao odgovarajući dvostruki integral.

**4.8.** Ako oblast $V$ nije ograničena ili ako funkcija $f(T)$ nije ograničena na $V$, definiše se nesvojstveni integral funkcije $f(T)$ po oblasti $V$ posve analogno kao u slučaju dvostrukog integrala.

# ZADACI

Izračunati integral:

**191.** $I=\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}.$

**192.** $I=\int_1^2 dz\int_0^1 dx\int_0^{xz}\left[2\,xy-\frac{y}{(z^2-y^2)^{3/2}}\right]dy.$

**193.** $I=\iiint\limits_V zdxdydz$, gdje je oblast

$$V:0\leqslant x\leqslant\frac{1}{2},\ \ x\leqslant y\leqslant 2\,x\ \text{ i }\ 0\leqslant z\leqslant\sqrt{1-x^2-y^2}.$$

**194.** $I=\iiint\limits_V xy^2z^3\,dxdydz$, gdje je oblast $V$ ograničena površima:

$$z=0,\quad x=1,\quad y=x\ \text{ i }\ z=xy.$$

**195.** $I=\iiint\limits_V zdxdydz$, gdje je oblast $V$ gornja polovina elipsoida

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1.$$

**196.** $I=\iiint\limits_V xydxdydz$, gdje je oblast $V$ ograničena površima:

$$az=y^2,\quad bz=y^2\ (y>0),$$

$$z=ax,\quad z=bx,\ z=h\,(a>0,\ b>0,\ h>0).$$

***Rješenja:***

**191.** $I=\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}=$

$$=\int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{-2(1+x+y+z)^2}\Bigg|_0^{1-x-y}\right]=$$

$$=\int_0^1 dx\int_0^{1-x}\frac{1}{2}\left[\frac{1}{(1+x+y)^2}-\frac{1}{4}\right]dy=$$

$$=\int_0^1\frac{1}{2}\left(\frac{1}{x+1}-\frac{3-x}{4}\right)dx=$$

$$=\frac{1}{2}\left(\ln|x+1|-\frac{3}{4}x+\frac{x^2}{8}\right)\Bigg|_0^1=\frac{1}{2}\left(\ln 2-\frac{5}{8}\right).$$

**192.** $I=\left(\frac{7}{12}-\frac{\pi}{2}\ln 2\right).$

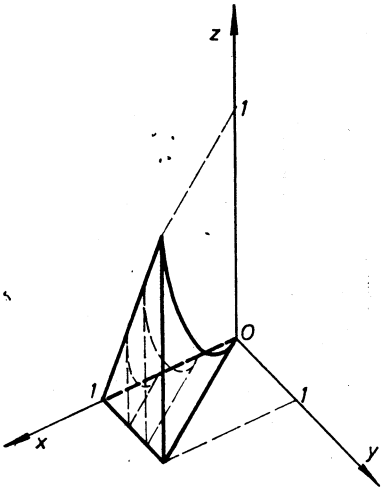
**193.** $I=\iiint_V z\,dx\,dy\,dz=\int_0^{\frac{1}{2}} dx\int_x^{2x} dy\int_0^{\sqrt{1-x^2-y^2}} z\,dz=$

$$=\int_0^{\frac{1}{2}} dx\int_x^{2x}\left[\frac{1}{2}z^2\Bigg|_0^{\sqrt{1-x^2-y^2}}\right]dy=\frac{1}{2}\int_0^{\frac{1}{2}} dx\int_x^{2x}(1-x^2-y^2)\,dy=$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}\left(y-x^2y-\frac{y^3}{3}\right)\Bigg|_x^{2x}dx=\frac{1}{2}\int_0^{\frac{1}{2}}\left(x-\frac{10}{3}x^3\right)dx=$$

$$=\frac{1}{2}\left(\frac{x^2}{2}-\frac{10}{12}x^4\right)\Bigg|_0^{\frac{1}{2}}=\frac{7}{192}.$$

**194.** Predstavimo oblast $V$ u koordinatnom sistemu (sl. 50) i uočimo granice:

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$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant x, \quad 0 \leqslant z \leqslant xy.$$

$$I = \iiint\limits_V xy^2 z^3 \, dx dy dz =$$

$$= \int_0^1 x \, dx \int_0^x y^2 \, dy \cdot \int_0^{xy} z^3 \, dz =$$

$$= \int_0^1 x dx \int_0^x y^2 \cdot \frac{x^4 y^4}{4} dy =$$

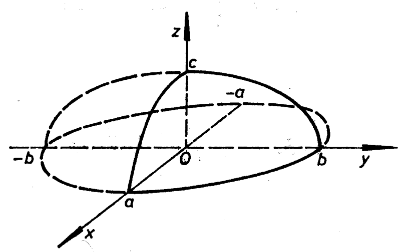
$$= \int_0^1 \frac{x^5}{4} dx \cdot \frac{y^7}{7} \Big|_0^x = \frac{x^{13}}{4 \cdot 7 \cdot 13} \Big|_0^1 = \frac{1}{364}.$$

**195.** $V$ je gornja polovina elipsoida $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$. (sl. 51).

Projekcija tijela na $xOy$-ravan je $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$. Znači: $-a \leqslant x \leqslant a$,

$$-\frac{b}{a}\sqrt{a^2 - x^2} \leqslant y \leqslant + \frac{b}{a}\sqrt{a^2 - x^2},$$

$$0 \leqslant z \leqslant c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$




Sl. 51

Imamo, dakle,

$$I = \iiint\limits_V z dx dy dz = \int_{-a}^{a} dx \int_{-\frac{b}{a}\sqrt{a^2 - x^2}}^{\frac{b}{a}\sqrt{a^2 - x^2}} dy \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} z dz =$$

$$= \frac{c^2}{2} \int_{-a}^{a} dx \int_{-\frac{b}{a}\sqrt{a^2 - x^2}}^{\frac{b}{a}\sqrt{a^2 - x^2}} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dy = \frac{2 bc^2}{3 a^3} \int_{-a}^{a} (a^2 - x^2)^{\frac{3}{2}} dx.$$

Ako uvedemo smjenu $x=a\sin t$, dobićemo

$$I=\frac{2bc^2}{3a^3}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a^4\cos^4 t\,dt=\frac{2abc^2}{3}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2 dt=$$

$$=\frac{2}{3}abc^2\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{4}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt=$$

$$=\frac{2}{3}abc^2\cdot\frac{1}{4}\left(\frac{3}{2}t+\sin 2t+\frac{\sin 4t}{8}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=abc^2\cdot\frac{\pi}{4}.$$

**196.** $I=\dfrac{h^4(b-a)^2(b+a)}{16\cdot a^2b^2}$.

Uvodeći (poopštene) cilindrične koordinate izračunati integrale:

**197.** $I=\displaystyle\iiint_V\frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, gdje je oblast integracije $V$:

$$x^2+y^2\leqslant 1,\quad -1\leqslant z\leqslant 1.$$

**198.** $I=\displaystyle\iiint_V(ax^2+by+cz^2)^3\,dxdydz$, ako je oblast $V$:

$$ax^2+cz^2\leqslant 1\ \ (a>0,\ c>0)\ \text{ i }\ 0\leqslant y\leqslant b.$$

**199.** $I=\displaystyle\iiint_V z\sqrt{x^2+y^2}\,dxdydz$, ukoliko je oblast $V$:

$$x^2+y^2\leqslant 2x,\quad y\geqslant 0,\quad 0\leqslant z\leqslant a:$$

**200.** $I=\displaystyle\iiint_V(x^2+y^2)\,dxdydz$, kad je oblast $V$ ograničena površima

$$z=2\ \text{ i }\ x^2+y^2=2z.$$

*Rješenja:*

**197.** Za cilindrične koordinate je $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$ i $|J|=\rho$. Pri tome će biti: $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant\rho\leqslant 1$, $-1\leqslant z\leqslant 1$. (sl. 52)

Dakle.

$$I=\iiint\limits_{V'}\frac{\rho\,d\rho\,d\varphi\,dz}{\sqrt{\rho^2+(z-2)^2}}=\int\limits_0^{2\pi}d\varphi\int\limits_{-1}^{1}dz\int\limits_0^1\frac{\rho\,d\rho}{\sqrt{\rho^2+(z-2)^2}}=$$

$$=2\pi\int\limits_{-1}^{1}\sqrt{\rho^2+(z-2)^2}\,\Big|_0^1\,dz=$$

$$=2\pi\int\limits_{-1}^{1}\left[\sqrt{1+(z-2)^2}-|z-2|\right]dz=$$

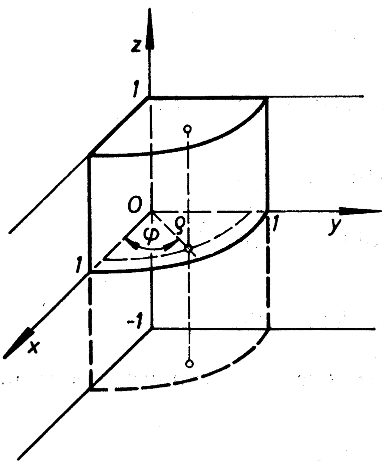
$$=2\pi\left[\int\limits_{-1}^{1}\sqrt{1+(z-2)^2}\,dz-\int\limits_{-1}^{1}(2-z)\,dz\right]=$$

(jer je $|z-2|=2-z$ za $-1\leqslant z\leqslant 1$)

$$=2\pi\left[\frac{z-2}{2}\sqrt{1+(z-2)^2}+\frac{1}{2}\ln\left(z-2+\sqrt{1+(z-2)^2}\right)-\right.$$

$$\left.-2z+\frac{z^2}{2}\right]_{-1}^{1}=2\pi\left[\frac{-\sqrt{2}}{2}+\frac{3}{2}\sqrt{10}-\frac{1}{2}\ln\frac{\sqrt{10}-3}{\sqrt{2}-1}-4\right].$$

**198.** Uvedimo poopštene cilindrične koordinate:

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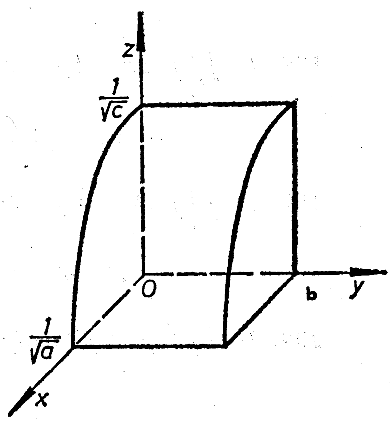


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$x=(\sqrt{a})^{-1}\rho\cos\varphi$, $y=y$, $z=(\sqrt{c})^{-1}\rho\sin\varphi$. Tada je $|J|=(\sqrt{a}\,\sqrt{c})^{-1}\cdot\rho$, a oblast $V':0\leqslant\varphi\leqslant 2\pi$, $0\leqslant\rho\leqslant 1$, $0\leqslant y\leqslant b$ (Oblast $V$ data je na sl. 53).

Prema tome,

$$I=\iiint\limits_{V'}(\rho^2+by)^3(\sqrt{ac})^{-1}\,d\rho\,d\varphi\,dy=$$

$$=(\sqrt{ac})^{-1}\int\limits_0^{2\pi}d\varphi\int\limits_0^1\rho\,d\rho\int\limits_0^b(\rho^2+by)^3\,dy=$$

$$=(\sqrt{ac})^{-1}\,2\pi\int\limits_0^1\rho\,d\rho\,\frac{(\rho^2+by)^4}{4b}\Bigg|_0^b=(\sqrt{ac})^{-1}\,\frac{2\pi}{4b}\int\limits_0^1[(\rho^2+b^2)^4-\rho^8]\rho\,d\rho=$$

$$=(\sqrt{ac})^{-1}\,\frac{\pi}{2b}\left[\frac{(\rho^2+b^2)^5}{10}-\frac{\rho^{10}}{10}\right]_0^b=(\sqrt{ac})^{-1}\,\frac{\pi}{20b}(32\,b^{10}-b^{10}-b^{10})=$$

$$=\frac{3}{2}b^9(\sqrt{ac})^{-1}\,\pi.$$

**199.** $I=\frac{8}{9}a^2.$

**200.** $I=\frac{16\pi}{3}.$

Uvodeći (generalisane) sferne koordinate izračunati:

**201.** $I=\iiint\limits_V(x^2+y^2)\,dxdydz$, gdje je $V$:

$$z\geqslant 0,\ r^2\leqslant x^2+y^2+z^2\leqslant R^2\quad(0<r<R).$$

**202.** $I=\iiint\limits_V(x+y+z)^2\,dxdydz$, gdje je $V$:

$$2\,az\geqslant x^2+y^2,\ x^2+y^2+z^2\leqslant 3\,a^2\quad(a>0).$$

**203.** $I=\iiint\limits_V\frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, gdje je $V$: $x^2+y^2+z^2\leqslant 1.$

**204.** $I=\iiint\limits_V\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz$, gdje je $V$: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1.$

*Rješenja:*

**201.** Oblast $V$ data je na sl. 54. Uvedimo sferne koordinate: $x=\rho\sin\theta\cos\varphi$, $y=\rho\sin\theta\sin\varphi$, $z=\rho\cos\theta$. Pri tome je $|J|=\rho^2\sin\theta$.

$$z\geqslant 0\Rightarrow\rho\cos\theta\geqslant 0\Rightarrow\cos\theta\geqslant 0\Rightarrow 0\leqslant\theta\leqslant\frac{\pi}{2},$$

$r^2\leqslant x^2+y^2+z^2\leqslant R^2\Rightarrow r^2\leqslant\rho^2\leqslant R^2\Rightarrow r\leqslant\rho\leqslant R$, dok je $0\leqslant\varphi\leqslant 2\pi$.

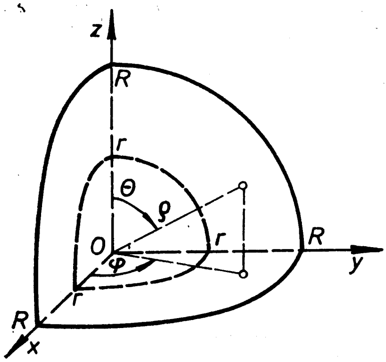
Prema tome,

$$I=\iiint_{V'} \rho^2 \sin^2\theta\cdot 1\cdot \rho^2 \sin\theta\, d\rho\, d\varphi\, d\theta =$$

$$=\int_0^{2\pi} d\varphi \int_0^{\pi/2} (1-\cos^2\theta)\sin\theta\, d\theta \int_r^R \rho^4\, d\rho =$$

$$=2\pi\left(1-\frac{1}{3}\right)\frac{R^5-r^5}{5}=\frac{4\pi}{15}(R^5-r^5).$$

**202.** U sfernim koordinatama jednačina sfere (sl. 55) biće:

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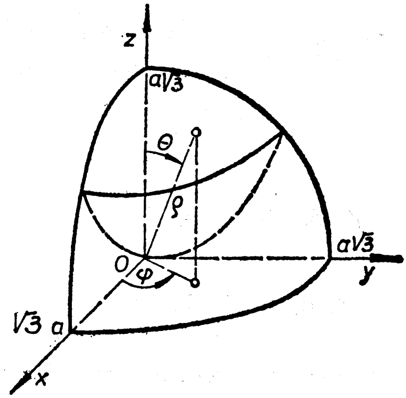



Sl. 55

$\rho=\sqrt{3}\,a$, a jednačina paraboloida:

$$2\,a\,\rho\cos\theta=\rho^2\sin^2\theta,\ \text{tj.}\ \rho=\frac{2\,a\cos\theta}{\sin^2\theta}.$$

Odredimo presjek sfere i paraboloida:

$$\sqrt{3}\,a=\frac{2\,a\cos\theta}{\sin^2\theta},$$

$$\sqrt{3}\cos^2\theta+2\cos\theta-\sqrt{3}=0,$$

$$\cos\theta=\frac{1}{\sqrt{3}},\quad \theta=\arccos\frac{1}{\sqrt{3}}.$$

Zato je

$$I=\iiint_{V'} \rho^2[(\cos\varphi+\sin\varphi)\sin\theta+\cos\theta]^2\cdot\rho^2\sin\theta\, d\rho\, d\varphi\, d\theta=$$

$$=\iiint_{V'}[(\cos^2\varphi+\sin^2\varphi+2\sin\varphi\cos\varphi)\sin^2\theta+2(\cos\varphi+\sin\varphi)\cdot$$

$$\cdot\sin\theta\cos\theta+\cos^2\theta]\,\rho^4\sin\theta\, d\rho\, d\varphi\, d\theta=$$

$$=\iiint_{V'}[1+2\sin\varphi\cos\varphi\sin^2\theta+2(\cos\varphi+\sin\varphi)\sin\theta\cos\theta]\cdot$$

$$\cdot\sin\theta\,\rho^4\, d\rho\, d\varphi\, d\theta.$$

$$I = \int\limits_0^{\arccos\frac{1}{\sqrt{3}}} d\theta \int\limits_0^{\sqrt{3}\,a} \rho^4 d\rho \int\limits_0^{2\pi} [1 + 2\sin\varphi\cos\varphi\sin^2\theta + 2(\cos\varphi + \sin\varphi)\cdot$$

$$\cdot\sin\theta\cos\theta]\sin\theta\, d\varphi + \int\limits_{\arccos\frac{1}{\sqrt{3}}}^{\pi/2} d\theta \int\limits_0^{\frac{2a\cos\theta}{\sin^2\theta}} \rho^4 d\rho \int\limits_0^{2\pi} [1 + 2\sin\varphi\cdot$$

$$\cdot\cos\varphi\sin^2\theta + 2(\cos\varphi + \sin\varphi)\sin\theta\cos\theta]\sin\theta\, d\varphi =$$

$$= 2\pi \int\limits_0^{\arccos\frac{1}{\sqrt{3}}} \sin\theta\, d\theta \int\limits_0^{\sqrt{3}\,a} \rho^4 d\rho + 2\pi \int\limits_{\arccos\frac{1}{\sqrt{3}}}^{\pi/2} \sin\theta\, d\theta \int\limits_0^{\frac{2a\cos\theta}{\sin^2\theta}} \rho^4 d\rho =$$

$$= 2\pi\frac{9\sqrt{3}\,a^5}{5}\left(1 - \frac{1}{\sqrt{3}}\right) + 2\pi \int\limits_{\arccos\frac{1}{\sqrt{3}}}^{\pi/2} \cdot\frac{32\,a^5}{5}\frac{\cos^5\theta}{\sin^9\theta}\, d\theta =$$

$$= \frac{18\sqrt{3}\,a^5\pi}{5} - \frac{18\,a^5\pi}{5} + \frac{64\pi a^5}{5}\,\frac{11}{3\cdot 8\cdot 16} = \frac{\pi a^5}{5}\left(18\sqrt{3} - \frac{97}{6}\right).$$

**203.** $I = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 d\rho \int\limits_0^{\pi} \frac{\rho^2 \sin\theta\, d\theta}{\sqrt{\rho^2 - 4\rho\cos\theta + 4}} = \frac{2\pi}{3}.$

**204.** Pomoću smjene $x = a\rho\cos\varphi\sin\theta$, $y = b\rho\sin\varphi\sin\theta$, $z = c\rho\cos\theta$, $|J| = abc\,\rho^2\sin\theta$, dobije se

$$I = \frac{\pi^2 abc}{4}.$$

Izračunati zapreminu oblasti ograničene datim površima:

**205.** $z = x^2 + y^2$, $z = 2x^2 + 2y^2$, $y = x$, $y = x^2$.

**206.** $2x^2 + z^2 = 4ax$, $2y^2 + z^2 = 2a^2$.

**207.** $z = \ln(x+2)$, $z = \ln(6-x)$, $x = 0$,

$$x + y = 2,\ x - y = 2.$$

**208.** $x = 0$, $z = 0$, $y = 1$, $y = 3$, $x + 2z = 3$.

**209.** $x^2 + y^2 + z^2 = R^2$. (Na šta se preslikava lopta $x^2 + y^2 + z^2 \leqslant R^2$ kad se uvedu cilindrične koordinate?)

**210.** $(a_1 x + b_1 y + c_1 z + d_1)^2 + (a_2 x + b_2 y + c_2 z + d_2)^2 +$
$+ (a_3 x + b_3 y + c_3 z + d_3)^2 = 1,$

pri čemu je $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$

**211.** $2z = x^2 + y^2, \ z = x + y.$

**212.** $x^2 + y^2 + z^2 = 4, \ x^2 + y^2 = 3z.$

**213.** $x^2 + y^2 - 2ax = 0, \ z = 0, \ x^2 + y^2 = z^2.$

**214.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

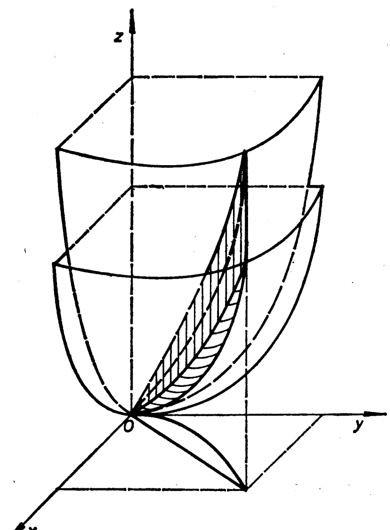
**215.** $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1 \ (a>0, \ b>0, \ c>0).$

**216.** $x^2 + y^2 + z^2 = 2Rx$ i $z^2 = x^2 + y^2 \quad (z \geqslant 0, \ z^2 \geqslant x^2 + y^2).$

**217.** $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$

**218.** $z = \frac{x^2 + y^2}{3}, \qquad z = \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 = 4$

$$\left(\frac{x^2 + y^2}{3} \leqslant z \leqslant \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 \leqslant 4\right).$$




Sl. 56

**219.** $(x^2 + y^2 + z^2)^2 = a^3 x.$

**220.** $(x^2 + y^2 + z^2)^3 = a^2 z^4.$

**221.** $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}.$

**222.** $\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)^2 = \frac{x}{h} - \frac{y}{k}$ $(x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0, \ h > 0, \ y > 0).$

**223.** Geometrijsko mjesto ortogonalnih projekcija koordinatnog početka na tangentne ravni elipsoida

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ (a>0, b>0, c>0; \ a=b).$

*Rješenja*:

**205.** Oblast $V$ data je na sl. 56.

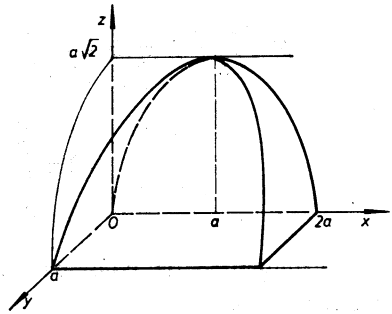
$$V = \iiint\limits_V dx\,dy\,dz = \int_0^1 dx \int_x^{x^2} dy \int_{x^2+y^2}^{2x^2+2y^2} dz = \frac{3}{35}. \ ?!$$

**206.** Date površi $2x^2+z^2=4ax$, $2y^2+z^2=2a^2$ možemo napisati u obliku

$$2(x-a)^2+z^2=2a^2$$

$$2y^2+z^2=2a^2.$$

One ograničavaju tijelo $V$ čija je jedna četvrtina prikazana na slici 57. Kako je



Sl. 57

$$x=a\pm\sqrt{a^2-\frac{z^2}{2}},\ \text{imamo}$$

$$V=\iiint\limits_V dx\,dy\,dz=$$

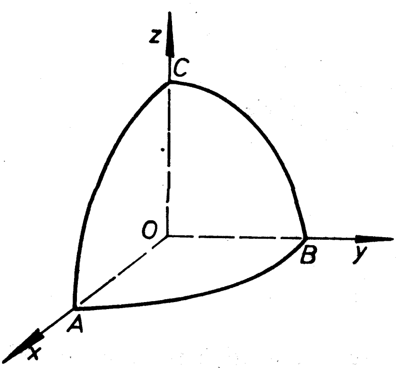
$$=4\int\limits_0^{\sqrt{2}a}dz\int\limits_0^{\sqrt{a^2-\frac{z^2}{2}}}dy\int\limits_{a-\sqrt{a^2-\frac{z^2}{2}}}^{a+\sqrt{a^2-\frac{z^2}{2}}}dx=4\int\limits_0^{\sqrt{2}a}2\sqrt{a^2-\frac{z^2}{2}}\cdot\sqrt{a^2-\frac{z^2}{2}}\,dz=$$

$$=8\int\limits_0^{\sqrt{2}a}\left(a^2-\frac{z^2}{2}\right)dz=\frac{16a^3\sqrt{2}}{3}.$$

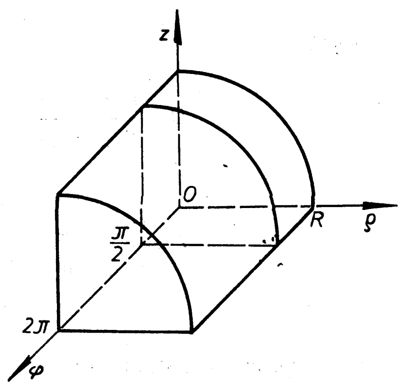
**207.** $V=4(4-3\ln 3)$.

**208.** $V=1$.

**209.** Ako uvedemo cilindrične koordinate $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$, nejednačina lopte (sl. 58) $x^2+y^2+z^2\leqslant R^2$ prelazi u $0\leqslant\varphi\leqslant 2\pi$, $\rho^2+z^2\leqslant R^2$, $(\rho\geqslant 0)$. U pravouglom koordinatnom sistemu $(\varphi,\ \rho,\ z)$ to predstavlja pola valjka (sl. 59).



Sl. 58



Sl. 59

Zapremina lopte biće

$$V=\iiint\limits_{V} dxdydz=\iiint\limits_{V'} \rho\, d\rho\, d\varphi\, dz=\int\limits_{0}^{2\pi} d\varphi \int\limits_{-R}^{R} dz \int\limits_{0}^{\sqrt{R^2-z^2}} \rho\, d\rho=$$

$$=2\pi \int\limits_{-R}^{R} \frac{1}{2}(R^2-z^2)\, dz=\frac{4R^3\pi}{3}.$$

Primijetimo da polovina valjka $V'$ na koju se preslikava lopta $V$ ima zapreminu

$$V=2\pi\cdot\frac{R^2\pi}{2}=R^2\pi^2,$$

koja, u opštem slučaju, nije jednaka zapremini lopte. Sjetimo se da je zapremina polovine valjka $V'$:

$$V'=\iiint\limits_{V'} d\varphi\, d\rho\, dz,$$

a zapremina lopte

$$V=\iiint\limits_{V} dxdydz=\iiint\limits_{V'} |J|\, d\varphi\, d\rho\, dz.$$

**210.** Uvedimo nove koordinate:

$$u=a_1x+b_1y+c_1z+d_1$$

$$v=a_2x+b_2y+c_2z+d_2$$

$$w=a_3x+b_3y+c_3z+d_3.$$

Pri tome je

$$D=\frac{D(u,v,w)}{D(x,y,z)}=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix},\ \text{dakle}$$

$$J=\frac{D(x,y,z)}{D(u,v,w)}=\frac{1}{\dfrac{D(u,v,w)}{D(x,y,z)}}=\frac{1}{D}.$$

Data površ u sistemu $(x, y, z)$ će se preslikati na sferu $u^2+v^2+w^2=1$ u sistemu $(u, v, w)$.

imaćemo

$$V=\iiint_V dxdydz=\iiint_{V'}\frac{1}{|D|}\,dudvdw=\frac{1}{|D|}\iiint_{V'}dudvdw=$$

$$=\frac{1}{|D|}\cdot\frac{4\cdot 1^3\pi}{3}=\frac{4\pi}{3|D|},\ \left(\text{jer je zapremina lopte } \frac{4R^3\pi}{3}\right).$$

**211.** Uvedimo cilindrične koordinate:

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=z,\ J=\rho.$$

Projekcija u $(x, y)$ ravni presječne linije paraboloida $2z=x^2+y^2$ i ravni $z=x+y$ biće $x^2+y^2=2(x+y)$ (sl. 60), tj. u cilindričnim koordinatama

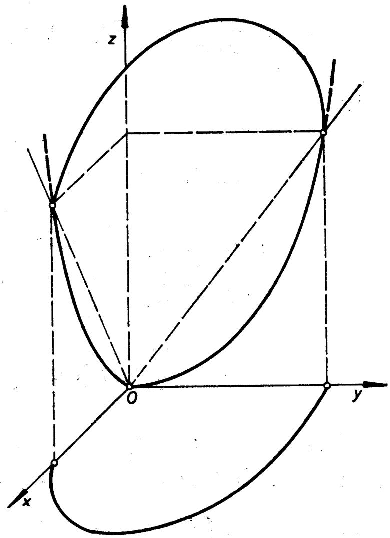
$$\rho=2(\cos\varphi+\sin\varphi).$$

Kako je $\rho\geqslant 0$, to mora biti

$$\cos\varphi+\sin\varphi\geqslant 0,\ \sin\left(\frac{\pi}{2}-\varphi\right)+\sin\varphi\geqslant 0,$$

$$2\sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}-\varphi\right)>0$$

$$-\frac{\pi}{2}\leqslant\frac{\pi}{4}-\varphi\leqslant\frac{\pi}{2},\ -\frac{\pi}{4}\leqslant\varphi\leqslant\frac{3\pi}{4}.$$




Sl. 60

Biće, dakle,

$$V=\iiint_V dxdydz=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}d\varphi\int_0^{2(\cos\varphi+\sin\varphi)}\rho\,d\rho\int_{\frac{\rho^2}{2}}^{\rho(\cos\varphi+\sin\varphi)}dz=\pi.$$

II način:

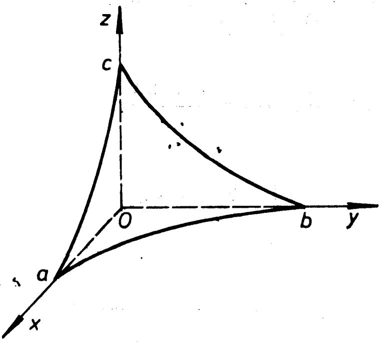
Kako je projekcija presječne linije paraboloida $2z=x^2+y^2$ i ravni $z=x+y$ kružnica $x^2+y^2=2(x+y)$, tj. $(x-1)^2+(y-1)^2=2$, to se mogu uvesti koordinate $x=1+\rho\cos\varphi$, $y=1+\rho\sin\varphi$, $z=z$, pa će biti $J=\rho$, a

$$V=\iiint_V dxdydz=\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}\rho\,d\rho\int_{\frac{2+2\rho(\cos\varphi+\sin\varphi)+\rho^2}{2}}^{2+\rho(\cos\varphi+\sin\varphi)}dz=\pi.$$

**212.** $V_1=\frac{19}{6}\pi,\qquad V_2=\frac{15}{2}\pi.$

**213.** $V=\frac{32}{9}a^3.\qquad$ **214.** $V=\frac{4}{3}abc\,\pi.$

**215.** Površ $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$ je simetrična u odnosu na koordinatne ravni (sl. 61 predstavlja dio te površi u I oktantu).




Sl. 61

Uvedimo smjenu:

$$\frac{x}{a}=X^3,\ \frac{y}{b}=Y^3,\ \frac{z}{c}=Z^3$$

i površ će se preslikati na loptu:

$$X^2+Y^2+Z^2=1.$$

Pri tome je

$$J=\frac{D(x,y,z)}{D(X,Y,Z)}=\begin{vmatrix} 3aX^2 & 0 & 0 \\ 0 & 3bY^2 & 0 \\ 0 & 0 & 3cZ^2 \end{vmatrix}=27\,abcX^2Y^2Z^2,\ \text{pa je}$$

$$V=\iiint\limits_V dxdydz=\iiint\limits_{V'} 27\,abcX^2Y^2Z^2\,dX\,dY\,dZ.$$

Uvedimo još jednom smjenu $X=\rho\sin\theta\cos\varphi$, $Y=\rho\sin\theta\sin\varphi$, $Z=\rho\cos\theta$. Sad je $J=\rho^2\sin\theta$, pa će biti

$$V=27\,abc\iiint\limits_{V''}\rho^2\sin^2\theta\cos^2\varphi\ \rho^2\sin^2\theta\sin^2\varphi\ \rho^2\cos^2\theta\cdot\rho^2\sin\theta\,d\rho\,d\varphi\,d\theta=$$

$$=27\,abc\int_0^1\rho^8\,d\rho\int_0^{2\pi}\sin^2\varphi\cos^2\varphi\,d\varphi\int_0^{\pi}\sin^4\theta\cos^2\theta\sin\theta\,d\theta=$$

$$=27\,abc\,\frac{\rho^9}{9}\Bigg|_0^1\int_0^{2\pi}\frac{1}{4}\sin^2 2\varphi\,d\varphi\int_0^{\pi}(1-2\cos^2\theta+\cos^4\theta)\cos^2\theta\sin\theta\,d\theta=$$

$$=27\,abc\cdot\frac{1}{9}\cdot\frac{1}{8}\left(\frac{2\varphi}{2}-\frac{\sin 4\varphi}{4}\right)\Bigg|_0^{2\pi}\left(-\frac{\cos^3\theta}{3}+2\frac{\cos^5\theta}{5}-\frac{\cos^7\theta}{7}\right)\Bigg|_0^{\pi}=$$

$$=\frac{3\,abc}{8}\cdot 2\pi\cdot 2\left(\frac{35-42+15}{3\cdot 5\cdot 7}\right)=\frac{4\,abc\,\pi}{35}.$$

**216.** Uvedimo sferne koordinate:

$$x=\rho\sin\theta\cos\varphi,\ \ y=\rho\sin\theta\sin\varphi,$$

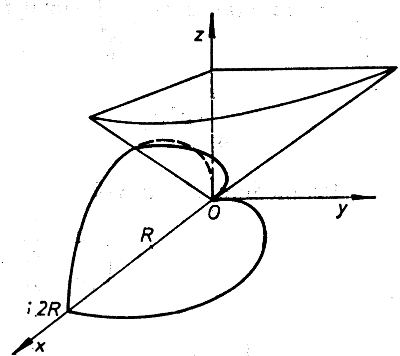
$$z=\rho\cos\theta,\ \ \text{i}\ \ J=\rho^2\sin\theta.$$

Konus $z^2 \leqslant x^2+y^2$ $(z \geqslant 0)$ određen je nejednačinom $\rho^2 \cos^2\theta \leqslant \rho^2 \sin^2\theta$, tj. $\operatorname{tg}^2\theta \leqslant 1$, odnosno $0 \leqslant \theta \leqslant \frac{\pi}{4}$, a lopta $x^2+y^2+z^2 \leqslant 2Rx$ nejednačinom $0 \leqslant \rho \leqslant 2R\sin\theta\cos\varphi$.

Oblast $V$ (sl. 62) preslikava se, dakle, na oblast $V'$ datu nejednačinama:

$$0 \leqslant \theta \leqslant \frac{\pi}{4}, \quad -\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}.$$

$$0 \leqslant \rho \leqslant 2R\sin\theta\cos\varphi.$$




Sl. 62

Zato je

$$V=\iiint\limits_{V} dx\,dy\,dz=\iiint\limits_{V'} \rho^2 \sin\theta\, d\rho\, d\varphi\, d\theta=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \int_0^{2R\sin\theta\cos\varphi} \rho^2\, d\rho=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{8R^3}{3}\sin^3\theta\cos^3\varphi\right]=\frac{8R^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi \int_0^{\frac{\pi}{4}} \sin^4\theta\, d\theta=$$

$$=\frac{8R^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\varphi)\cos\varphi\, d\varphi \cdot \int_0^{\frac{\pi}{4}} \frac{1}{4}\left(1-2\cos 2\theta+\frac{1-\cos 4\theta}{2}\right) d\theta=$$

$$=\frac{R^3}{9}(3\pi-8).$$

**217.** Odmah se vidi da je tijelo simetrično u odnosu na koordinatne ravni. Da bismo predstavili tijelo u koordinatnom sistemu $Oxyz$, pređimo na uopštene sferne koordinate:

$$x=a\rho\cos\varphi\sin\theta, \quad y=b\rho\sin\varphi\sin\theta,$$

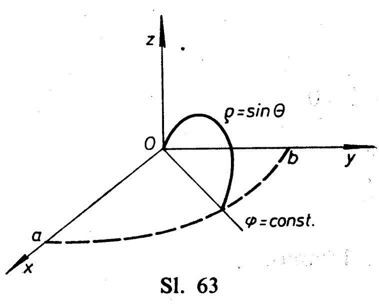
$$z=c\rho\cos\theta \text{ i } J=abc\rho^2\sin\theta.$$

Imaćemo

$$\rho^4=\rho^2\sin^2\theta, \text{ tj. } \rho=\sin\theta.$$

U ravni $\varphi=\text{const.}$ imamo krivu $\rho=\sin\theta$, a to je elipsa (sl. 63).

Rotacijom te elipse oko $z$-ose dobija se površ koja ograničava tijelo $V$.




Sl. 63

Zapremina tijela $V$ je

$$V=\iiint\limits_V dxdydz = abc\iiint\limits_{V'} \rho^2\sin\theta\cdot d\rho\, d\varphi\, d\theta = abc\int\limits_0^{2\pi} d\varphi\int\limits_0^{\pi}\sin\theta\, d\theta\cdot$$

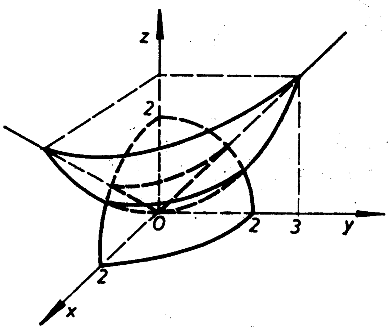
$$\cdot\int\limits_0^{\sin\theta}\rho^2\,d\rho = abc\,2\pi\cdot\frac{1}{3}\int\limits_0^{\pi}\sin^4\theta\, d\theta = abc\frac{2\pi}{3}\int\limits_0^{\pi}\left(\frac{3}{8}-\frac{\cos 2\theta}{2}-\frac{\cos 4\theta}{8}\right)d\theta=$$

$$=\frac{\pi^2}{4}\cdot abc.$$

Primjedba: Za površ

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

dobija se očigledno isti rezultat $\frac{\pi^2}{4}abc.$ Koju od koordinatnih osa treba sada uzeti kao osu sfernog koordinatnog sistema?



Sl. 64

**218.** Tijelo ograničavaju sfera $x^2+y^2+z^2=4$, paraboloid $z=\frac{x^2+y^2}{3}$ i konus $z=\sqrt{x^2+y^2}$ (sl. 64). Jednačine tih površi u sfernim koordinatama biće respektivno:

$$\rho=2,\quad \rho=\frac{3\cos\theta}{\sin^2\theta},\quad \theta=\frac{\pi}{4}.$$

Odredimo presjek lopte i paraboloida.

$$2=\frac{3\cos\theta}{\sin^2\theta}$$

$$2\sin^2\theta=3\cos\theta$$

$$2\cos^2\theta+3\cos\theta-2=0$$

$$\cos\theta=\frac{1}{2},\quad \theta=\frac{\pi}{3}.$$

$\rho=2,\ \theta=\frac{\pi}{3}$ je presjek lopte i paraboloida. Imamo:

$$V=\iiint\limits_{V} dx\,dy\,dz=\iiint\limits_{V'} \rho^2 \sin\theta\, d\rho\, d\varphi\, d\theta=$$

$$=\int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta \int_2^{\frac{3\cos\theta}{\sin^2\theta}} \rho^2 \sin\theta\, d\rho=$$

$$=2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{3}\left[\frac{27\cos^3\theta}{\sin^6\theta}-8\right]\sin\theta\, d\theta=$$

$$=\frac{2\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left[\frac{27(1-\sin^2\theta)\,2\sin\theta\cos\theta}{2(\sin^2\theta)^3}-8\sin\theta\right]d\theta=$$

$$=\frac{2\pi}{3}\,\frac{27}{2}\left[\frac{1}{-2(\sin^2\theta)^2}-\frac{1}{-\sin^2\theta}+\frac{2\cdot 8}{27}\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{2\pi}{3}\left(10-4\sqrt{2}\right).$$

**219.** $V=\dfrac{1}{3}\pi a^3.$ (Koristiti sferne koordinate. Oblast $V'$ data je nejednačinama:

$$-\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2},\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\rho\leqslant\sqrt[3]{a^3\sin\theta\cos\varphi}\Big).$$

**220.** $V=\dfrac{4}{21}\pi a^3.$

**221.** $V=\dfrac{abc}{4\sqrt{2}}\cdot\pi^2.$

**222.** $V=\dfrac{abc}{60}\,\dfrac{\left(\dfrac{a}{h}\right)^4}{\dfrac{a}{h}+\dfrac{b}{k}}.$

**223.** Uzmimo tačku $(x, y, z)$ na elipsoidu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{1}$$

Tada jednačina tangentne ravni glasi

$$\frac{xX}{a^2}+\frac{yY}{b^2}+\frac{zZ}{c^2}=1. \tag{2}$$

Jednačina normale iz (0, 0, 0) na tangentnu ravan glasi

$$\frac{X}{\frac{x}{a^2}}=\frac{Y}{\frac{y}{b^2}}=\frac{Z}{\frac{z}{c^2}}. \tag{3}$$

Iz jednačina (1), (2) i (3) eliminišimo $x$, $y$ i $z$.

Iz (3) je:

$$y=\frac{b^2 x}{a^2}\,\frac{Y}{X},$$

$$z=\frac{c^2 x}{a^2}\,\frac{Z}{X},$$

pa kad to uvrstimo u jednačinu (2), dobijamo

$$x\frac{X}{a^2}+\frac{x}{a^2}\,\frac{Y^2}{X}+\frac{x}{a^2}\,\frac{Z^2}{X}=1,$$

tj.

$$x=\frac{a^2 X}{X^2+Y^2+Z^2},\quad y=\frac{b^2 Y}{X^2+Y^2+Z^2},\quad z=\frac{c^2 Z}{X^2+Y^2+Z^2}.$$

Kad ove vrijednosti za $x, y, z$ uvrstimo u jednačinu (1) dobićemo:

$$\frac{\left(\frac{a^2 X}{X^2+Y^2+Z^2}\right)^2}{a^2}+\frac{\left(\frac{b^2 Y}{X^2+Y^2+Z^2}\right)^2}{b^2}+\frac{\left(\frac{c^2 Z}{X^2+Y^2+Z^2}\right)^2}{c^2}=1,$$

odnosno $a^2 X^2+b^2 Y^2+c^2 Z^2=(X^2+Y^2+Z^2)^2$.

Nađena površ je simetrična u odnosu na koordinatne ravni.

b) Ako je $a=b$, imamo:

$$(X^2+Y^2+Z^2)^2=a^2(X^2+Y^2)+c^2 Z^2.$$

Uvedimo sferne koordinate:

$$\rho^4=a^2\rho^2\sin^2\theta+c^2\rho^2\cos^2\theta$$

$$\rho^2=a^2-(a^2-c^2)\cos^2\theta.$$

Zapremina tijela biće:

$$V=\iiint_V dX\,dY\,dZ=8\int_0^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\int_0^{\sqrt{a^2-(a^2-c^2)\cos^2\theta}}\rho^2\sin\theta\,d\rho=$$

$$=8\,\frac{\pi}{2}\cdot\frac{1}{3}\int_0^{\frac{\pi}{2}}\left[\sqrt{a^2-(a^2-c^2)\cos^2\theta}\right]^3\sin\theta\,d\theta.$$

Uvodimo smjenu:

$$\sqrt{a^2-c^2}\cos\theta=t, \qquad -\sin\theta\, d\theta=\frac{dt}{\sqrt{a^2-c^2}}.$$

$$V=\frac{4\pi}{3}\frac{1}{\sqrt{a^2-c^2}}\int_0^{\sqrt{a^2-c^2}}\left(\sqrt{a^2-t^2}\right)^3 dt=$$

$$=\frac{4\pi}{3}\frac{1}{\sqrt{a^2-c^2}}\left[\left(-\frac{1}{4}t^3+\frac{5}{8}a^2t\right)\sqrt{a^2-t^2}+\frac{3}{8}a^4\arcsin\frac{t}{a}\right]_0^{\sqrt{a^2-c^2}}=$$

$$=\frac{4\pi}{3}\frac{1}{\sqrt{a^2-c^2}}\left[\sqrt{a^2-c^2}\left(-\frac{a^2-c^2}{4}+\frac{5}{8}a^2\right)\cdot c+\frac{3a^4}{8}\arcsin\frac{\sqrt{a^2-c^2}}{a}\right].$$

(Specijalno, ako pređemo na limes kad $a\to c$, dobićemo zapreminu lopte $\frac{4\pi c^3}{3}$.)

Odrediti težište tijela $V$, ako je:

**224.** Tijelo $V$ homogeno i ograničeno površima $x^2+y^2+z^2=a^2$, $x^2+y^2=ax$.

**225.** $V$ zajednički dio lopti $x^2+y^2+z^2\leqslant R^2$, $x^2+y^2+z^2\leqslant 2Rz$, a gustina u svakoj tački tijela brojno jednaka rastojanju te tačke od ravni $xOy$.

**226.** $V$ lopta $x^2+y^2+z^2\leqslant 2Rz$, a gustina u bilo kojoj tački jednaka kvadratu rastojanja te tačke od koordinatnog početka.

Naći težište homogenog tijela ograničenog površima:

**227.** $x^2=2pz$, $y^2=2px$, $x=\frac{p}{2}$, $z=0$.

**228.** $x^2+y^2=2z$, $x+y=z$.

**229.** $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{xyz}{abc}$ $(x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0)$.

Odrediti moment inercije tijela $V$ u odnosu na osu $\vec{p}$, ako je:

**230.** $V$ homogeno tijelo ograničeno površima $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$, $\frac{x}{a}+\frac{y}{b}=\frac{z}{c}$, a $\vec{p}$ $z$-osa.

**231.** $V$ homogeno tijelo ograničeno površima $x^2+y^2+z^2=R^2$, $x^2+y^2=z^2$ $(z\geqslant 0)$, a $\vec{p}$ $z$-osa.

**232.** $V$ lopta $x^2+y^2+z^2\leqslant R^2$, a gustina u svakoj tački proporcionalna rastojanju te tačke od centra lopte i osa dijametar lopte.

**233.** $V$ homogeni elipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$ a $\vec{p}$ $z$-osa.

**234.** $V$ homogeno tijelo određeno nejednačinama $0\leqslant z\leqslant x^2+y^2$, $-1\leqslant x+y\leqslant 1$, $-1\leqslant x-y\leqslant 1$, a $\vec{p}$ $z$-osa.

**235.** Odrediti moment inercije u odnosu na koordinatni početak homogenog tijela ograničenog površi $(x^2+y^2+z^2)^2=a^2(x^2+y^2)$.

**236.** Naći moment inercije homogenog torusa: $x=(a+b\cos\theta)\cos\varphi$, $y=(a+b\cos\theta)\sin\varphi$, $z=b\sin\theta$ u odnosu na $z$-osu $(a>b>0)$.

**237.** Izračunati moment inercije homogenog torusa u odnosu na $x$-osu.

Naći silu kojom tijelo $V$ privlači materijalnu tačku mase $m$, ako je:

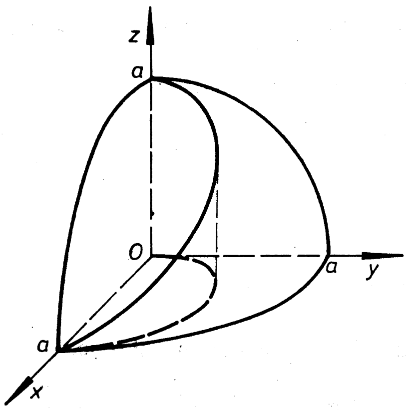
**238.** $V$ homogena lopta $x^2+y^2+z^2\leqslant R^2$, a materijalna tačka udaljena od centra lopte za $a$.

**239.** $V$ lopta $x^2+y^2+z^2\leqslant R^2$ sa gustoćom mase $\rho=z^2$, a masa $m$ se nalazi u tački $A(0,\ 0,\ 2R)$.

**240.** Dato je homogeno tijelo ograničeno sa dvije koncentrične sfere. Dokazati da je privlačna sila tijela na materijalnu tačku koja se nalazi unutar tijela jednaka 0.

**241.** Naći silu kojom homogeni cilindar $x^2+y^2\leqslant R^2$, $0\leqslant z\leqslant h$ privlači jediničnu masu smještenu u centru donje osnove cilindra.

*Rješenja:*

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**224.** Zbog simetrije u odnosu na $x$-osu je $y_T=z_T=0$. Odredimo

$$x_T=\frac{\iiint\limits_V x\,dx\,dy\,dz}{\iiint\limits_V dx\,dy\,dz}.$$

Tijelo $V$ projicira se na domen $D$ (sl. 65), koji je u polarnim koordinatama određen nejednakostima

$$\rho\leqslant a\cos\varphi,\ -\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2}.$$

$$\iiint\limits_V dx\,dy\,dz=\iint\limits_D 2\sqrt{a^2-(x^2+y^2)}\,dx\,dy=\int\limits_{-\pi/2}^{\pi/2}d\varphi\int\limits_0^{a\cos\varphi}2\sqrt{a^2-\rho^2}\,\rho\,d\rho=$$

$$=\int\limits_{-\pi/2}^{\pi/2}\left(-\frac{2}{3}a^3|\sin^3\varphi|+\frac{2}{3}a^3\right)d\varphi=\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right);$$

$$\iiint\limits_V x\,dx\,dy\,dz = \iint\limits_D x\cdot 2\sqrt{a^2-(x^2+y^2)}\,dx\,dy = 2\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \int\limits_0^{a\cos\varphi} \sqrt{a^2-\rho^2}\cdot\rho^2\,d\rho\,d\varphi =$$

$$= (\rho = a\cos t,\ d\rho = -a\sin t\,dt) =$$

$$= 2\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi \int\limits_{\frac{\pi}{2}}^{\varphi} -a\sin t\,a^2\cos^2 t\,a\sin t\,dt =$$

$$= -\frac{2a^4}{4}\int\limits_{-\frac{\pi}{2}}^{\pi/2}\cos\varphi\,d\varphi \int\limits_{\frac{\pi}{4}}^{\varphi}\sin^2 2t\,dt =$$

$$= -\frac{2a^4}{2\cdot 4}\int\limits_{-\pi/2}^{\pi/2}\cos\varphi\left[\frac{2t}{2}-\frac{\sin 4t}{4}\right]_{\pi/2}^{\varphi} d\varphi =$$

$$= -\frac{2a^4}{2\cdot 4}\int\limits_{-\pi/2}^{\pi/2}\cos\varphi\left[\varphi-\frac{\sin 4\varphi}{4}-\frac{\pi}{2}\right]d\varphi =$$

$$= -\frac{a^4}{4}\left[0-\frac{1}{4}\int\limits_{-\pi/2}^{\pi/2}\sin 4\varphi\cos\varphi\,d\varphi-\frac{\pi}{2}\sin\varphi\Big|_{-\pi/2}^{\pi/2}\right]=$$

$$= \frac{a^4\pi}{8}\cdot\sin\varphi\Big|_{-\pi/2}^{\pi/2} = \frac{a^4\pi}{4}.$$

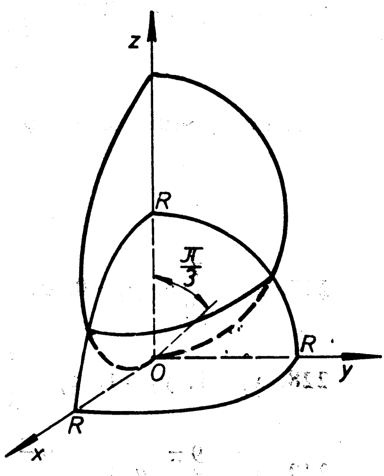
Dakle,

$$x_T = \frac{\dfrac{a^4\pi}{4}}{\dfrac{2}{3}a^3\left(\pi-\dfrac{4}{3}\right)} = \frac{9a\pi}{8(3\pi-4)}.$$

**225.** Odmah se vidi da je $x_T = y_T = 0$ (zbog simetrije tijela (sl. 66) i gustoće u odnosu na $z$-osu).

Ostaje da izračunamo

$$z_T = \frac{\iiint\limits_V z\cdot z\,dx\,dy\,dz}{\iiint\limits_V z\,dx\,dy\,dz}.$$



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Uvedimo sferne koordinate: $x=\rho\sin\theta\cos\varphi$, $y=\rho\sin\theta\sin\varphi$, $z=\rho\cos\theta$, $J=\rho^2\sin\theta$.

Jednačine graničnih površi glase: $\rho=R$, $\rho=2R\cos\theta$, pa je presjek tih površi: $2\cos\theta=1$, $\cos\theta=\frac{1}{2}$, tj. $\theta=\frac{\pi}{3}$. Zato je

$$\iiint_V z\,dxdydz=\iiint_{V'}\rho\cos\theta\cdot\rho^2\sin\theta\,d\rho\,d\varphi\,d\theta=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi/3}\sin\theta\cos\theta\,d\theta\int_0^R\rho^3\,d\rho+\int_0^{2\pi}d\varphi\int_{\pi/3}^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{2R\cos\theta}\rho^3\,d\rho=$$

$$=2\pi\cdot\frac{1}{2}\,\frac{3}{4}\,\frac{R^4}{4}+2\pi\cdot 4R^4\frac{1}{6}\cdot\frac{1}{64}=R^4\pi\cdot\frac{5}{24};$$

$$\iiint_V z^2\,dxdydz=\iiint_{V'}\rho^4\cos^2\theta\sin\theta\,d\rho\,d\varphi\,d\theta=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi/3}\cos^2\theta\sin\theta\,d\theta\int_0^R\rho^4\,d\rho+\int_0^{2\pi}d\varphi\int_{\pi/3}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\int_0^{2R\cos\theta}\rho^4\,d\rho=$$

$$=2\pi\cdot\frac{1}{3}\,\frac{7}{8}\cdot\frac{R^5}{5}+2\pi\cdot\frac{32R^5}{5}\cdot\frac{1}{8}\cdot\frac{1}{2^8}=\frac{59R^5}{15\cdot 32}\pi.$$

Prema tome,

$$z_T=\frac{59}{100}R.$$

**226.** $x_T=0$; $y_T=0$; $z_T=\dfrac{\iiint_V z(x^2+y^2+z^2)\,dxdydz}{\iiint_V (x^2+y^2+z^2)\,dxdydz}$.

Uvesti sferne koordinate: $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant\theta\leqslant\frac{\pi}{2}$, $0\leqslant\rho\leqslant 2R\cos\theta$;

$$z_T=\frac{5}{4}R.$$

**227.** $x_T=\frac{7}{18}p$, $y_T=0$, $z_T=\frac{7}{196}p$.

**228.** $x_T=1$, $y_T=1$, $z_T=\frac{5}{3}$.

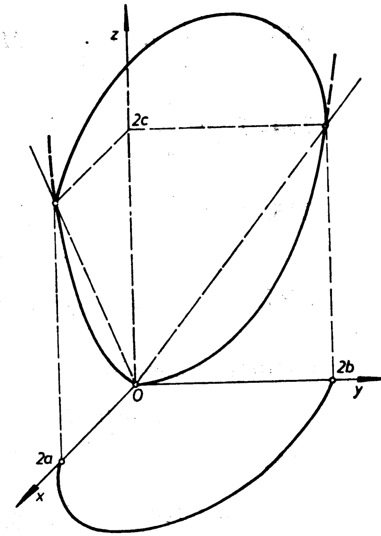
**229.** $x_T=\frac{9\pi}{448}a$, $y_T=\frac{9\pi}{448}b$, $z_T=\frac{9\pi}{448}c$.

**230.** Tijelo (sl. 67) ograničavaju paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$ i ravan $\frac{x}{a}+\frac{y}{b}=\frac{z}{c}$.

Projekcija presječne linije tih površi na $xOy$-ravan biće

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a}+\frac{2y}{b}, \text{ tj.}$$

$$\left(\frac{x}{a}-1\right)^2+\left(\frac{y}{b}-1\right)^2=2.$$




Sl. 67

Moment inercije

$$I_z=\gamma\iiint_V (x^2+y^2)\,dxdydz=$$

$$=\gamma\iint_D (x^2+y^2)\left[c\left(\frac{x}{a}+\frac{y}{b}\right)-\frac{c}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right]dxdy=$$

$$=\frac{\gamma c}{2}\iint_D (x^2+y^2)\left[2-\left(\frac{x}{a}-1\right)^2-\left(\frac{y}{b}-1\right)^2\right]dxdy.$$

Uvedimo koordinate: $\frac{x}{a}-1=\rho\cos\varphi$, $\frac{y}{b}-1=\rho\sin\varphi$; $J=ab\rho$.

Presječna linija imaće jednačinu $\rho=\sqrt{2}$.

$$I_z=\frac{\gamma cab}{2}\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}[a^2(1+\rho\cos\varphi)^2+b^2(1+\rho\sin\varphi)^2][2-\rho^2]\cdot\rho\,d\rho=$$

$$=\frac{\gamma cab}{2}\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}[a^2+b^2)(2\rho-\rho^3)+(2a^2\cos\varphi+2b^2\sin\varphi)(2\rho^2-\rho^4)+$$
$$+(a^2\cos^2\varphi+b^2\sin^2\varphi)(2\rho^3-\rho^5)]\,d\rho=$$

$$=\frac{\gamma abc}{2}\left[(a^2+b^2)\cdot 2\pi(2-1)+0+(a^2\pi+b^2\pi)\cdot\left(2-\frac{4}{3}\right)\right]=$$

$$=\gamma\frac{4}{3}abc(a^2+b^2)\pi.$$

**231.** $I_z=\gamma\,\frac{8-5\sqrt{2}}{30}$ (uvesti sferne koordinate).

**232.** $I_z = \frac{4}{9} MR^2$ ($M$ masa lopte).

**233.** $I_z = \frac{1}{5} M(a^2 + b^2)$ ($M$ masa elipsoida).

**234.** $I_z = \gamma \frac{14}{45}$.

**235.** $I_0 = \gamma \frac{\pi^2 a^5}{8}$.

**236.** $I_z = \gamma \iiint\limits_V (x^2 + y^2)\, dxdydz$.

Uvedimo koordinate: $x = (a + \rho \cos\theta)\cos\varphi$, $y = (a + \rho\cos\theta)\sin\varphi$,

$$z = \rho \sin\theta \quad (0 \leqslant \rho \leqslant b;\ 0 \leqslant \varphi \leqslant 2\pi;\ 0 \leqslant \theta \leqslant 2\pi).$$

Tada je

$$J = \begin{vmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi & -(a+\rho\cos\theta)\sin\varphi \\ \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & (a+\rho\cos\theta)\cos\varphi \\ \sin\theta & \rho\cos\theta & 0 \end{vmatrix}, \text{ tj.}$$

$|J| = \rho(a + \rho\cos\theta)$.

$$I_z = \gamma \iiint\limits_{V'} (a + \rho\cos\theta)^2 \cdot (a + \rho\cos\theta)\, \rho\, d\rho\, d\varphi\, d\theta =$$

$$= \gamma \int_0^{2\pi} d\varphi \int_0^{2\pi} d\theta \int_0^b (a + \rho\cos\theta)^3 \rho\, d\rho =$$

$$= \gamma \cdot 2\pi \int_0^{2\pi} \left[ a^3 \frac{b^2}{2} + 3a^2\cos\theta \frac{b^3}{3} + 3a\cos^2\theta \frac{b^4}{4} + \cos^3\theta \frac{b^5}{5} \right] d\theta =$$

$$= \gamma \cdot 2\pi \left[ \frac{a^3 b^2}{2} \cdot 2\pi + 0 + \frac{3a b^4}{4} \cdot \pi + 0 \right] = \frac{\gamma\pi^2}{2} ab^2 (4a^2 + 3b^2).$$

**237.** $I_x = \gamma \frac{\pi^2}{4} ab^2 (4a^2 + 5b^2)$.

**238.** Neka je masa $m$ na $z$-osi. Tada homogena lopta $V$ privlači masu $m$ silom $F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$, pri čemu je $F_x = F_y = 0$ (zbog simetrije), dok je

$$F_z = m\gamma \iiint\limits_V \frac{(z-a)\, dxdydz}{[x^2 + y^2 + (z-a)^2]^{3/2}}.$$

Uvedimo cilindrične koordinate: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$, $|J| = \rho$. Lopta će biti zadana nejednačinom $z^2 + \rho^2 \leqslant R^2$, pa je

$$F_z = m\gamma \int_0^{2\pi} d\varphi \int_{-R}^{R} (z-a)\, dz \int_0^{\sqrt{R^2-z^2}} \frac{\rho\, d\rho}{[\rho^2 + (z-a)^2]^{3/2}} =$$

$$= m\gamma \cdot 2\pi \int_{-R}^{R} \left[\frac{-1}{\sqrt{\rho^2 + (z-a)^2}}\right]_0^{\sqrt{R^2-z^2}} (z-a)\, dz =$$

$$= m\gamma\, 2\pi \int_{-R}^{R} \left(\frac{z-a}{|z-a|} - \frac{z-a}{\sqrt{R^2 - 2az + a^2}}\right) dz.$$

Za $a \leqslant R$ imamo

$$\int_{-R}^{R} \frac{z-a}{|z-a|}\, dz = \int_{-R}^{R} \operatorname{sgn}(z-a)\, dz = \int_{-R}^{a} -dz + \int_{a}^{R} dz = -a - R + R - a = -2a,$$

dok za $a > R$ vrijedi

$$\int_{-R}^{R} \frac{z-a}{|z-a|}\, dz = \int_{-R}^{R} \operatorname{sgn}(z-a)\, dz = \int_{-R}^{R} -dz = -2R.$$

Drugi integral biće:

$$I_2 = \int_{-R}^{R} \frac{z-a}{\sqrt{R^2 - 2az + a^2}}\, dz = \left[R^2 - 2az + a^2 = t^2,\ z = \frac{R^2 + a^2 - t^2}{2a},\ dz = -\frac{t\, dt}{a}\right] =$$

$$= \int_{R+a}^{|R-a|} \left(\frac{R^2 + a^2 - t^2}{2a} - a\right) \frac{-dt}{a} = \frac{1}{2a^2} \int_{|R-a|}^{R+a} (R^2 - a^2 - t^2)\, dt =$$

$$= \frac{1}{2a^2} \left[(R^2 - a^2)t - \frac{t^3}{3}\right]_{|R-a|}^{R+a} =$$

$$= \frac{1}{2a^2} \left[(R+a)\left(R^2 - a^2 - \frac{R^2 + 2aR + a^2}{3}\right) - |R-a|\left(R^2 - a^2 - \frac{R^2 - 2aR + a^2}{3}\right)\right] =$$

$$= \frac{1}{6a^2} [(R+a)(2R^2 - 2aR - 4a^2) - |R-a|(2R^2 + 2aR - 4a^2)].$$

Za $a \leqslant R$ biće:

$$I_2 = \frac{1}{6a^2}[(2R^3 - 2aR^2 - 4a^2 R + 2aR^2 - 2a^2 R - 4a^3) -$$

$$- (2R^3 + 2aR^2 - 4a^2 R - 2aR^2 - 2a^2 R + 4a^3)] = -\frac{4a}{3},$$

a za $a > R$

$$I_2 = \frac{1}{6a^2}[(2R^3 - 2aR^2 - 4a^2 R + 2aR^2 - 2a^2 R - 4a^3) +$$

$$+ (2R^3 + 2aR^2 - 4a^2 R - 2aR^2 - 2a^2 R + 4a^3)] = \frac{2R^3}{3a^2} - 2R.$$

Na osnovu toga imamo:

$$F_z = \begin{cases} m\gamma \cdot \dfrac{-4a\pi}{3}, & \text{za} \quad a \leqslant R \\ m\gamma \cdot \dfrac{-4R^3\pi}{3a^2}, & \text{za} \quad a > R. \end{cases}$$

Vidi se da sila privlačenja djeluje prema centru lopte.

Pri tome, ako se masa $m$ nalazi van sfere ($a > R$), može se vidjeti da je privlačna sila upravo jednaka privlačnoj sili mase $M$ lopte skoncentrisane u centru te lopte:

$$F = \frac{\gamma \cdot \dfrac{4R^3\pi}{3} \cdot m}{a^2}.$$

S druge strane, ako se masa $m$ nalazi u lopti ($a \leqslant R$), privlačna sila ne zavisi od $R$, nego samo od udaljenosti $a$ materijalne tačke od centra lopte i ima istu veličinu kao u gornjem slučaju kad je $R = a$.

**239.** $F_x = 0, \quad F_y = 0.$

$$F_z = m\gamma \iiint\limits_V \frac{z^2(z-2R)\,dxdydz}{[x^2 + y^2 + (z-2R)^2]^{3/2}} =$$

$$= m\gamma \int_0^{2\pi} d\varphi \int_{-R}^{R} z^2(z-2R) \int_0^{\sqrt{R^2-z^2}} \frac{\rho\, d\rho}{[\rho^2 + (z-2R)^2]^{3/2}} =$$

$$= m\gamma\, 2\pi \int_{-R}^{R} z^2(z-2R)\left[-\frac{1}{\sqrt{5R^2 - 4Rz}} + \frac{1}{|z-2R|}\right] dz =$$

$$= m\gamma\, 2\pi \left[\int_{-R}^{R} -z^2\, dz - \int_{-R}^{R} \frac{z^2(z-2R)}{\sqrt{5R^2 - 4Rz}}\, dz\right].$$

Imamo

$$I_1 = \int_{-R}^{R} \frac{z^2(z-2R)}{\sqrt{5R^2-4Rz}}\,dz = [5R^2-4Rz = R^2t^2, \quad -4R\,dz = 2R^2\,t\,dt] =$$

$$= \int_{3}^{1} \frac{\left[\frac{R(5-t^2)}{4}\right]^2\left[\frac{R(5-t^2)}{4}-2R\right]}{Rt}\,\frac{-Rt}{2}\,dt =$$

$$= \frac{R^3}{128}\int_{1}^{3}(25-10t^2+t^4)(-3-t^2)\,dt =$$

$$= -\frac{R^3}{128}\left(\frac{t^7}{7}-7\frac{t^5}{5}-5\frac{t^3}{3}+75t\right)\Bigg|_1^3 = -\frac{R^3}{128}\,\frac{8416}{105}.$$

Prema tome,

$$F_z = m\gamma\,2\pi\left[-\frac{2R^3}{3}+\frac{R^3}{4}\,\frac{263}{105}\right] = -m\gamma\,\frac{17\,R^3\pi}{210}.$$

**240.** Koristiti rješenje zadatka 236. i 239. $F_z = \gamma\,2\pi(R+h+\sqrt{R^2+h^2})$.